**Introduction**

**Dirichlet Kernel**

**Derivation of the Dirichlet kernel**

First, we recall that the complex form of the Fourier series is $$ f(x) = \sum^{\infty}\_{n = -\infty} c\_n e^{inx} $$. We can rewrite this as an N-th partial sum $$ S\_Nf= \sum^{N}\_{n = -N} c\_n e^{inx} $$. We should also remember that for some function that is absolutely integrable over the interval $$ [-\pi, \pi] $$, the definition of its coefficients is $$ c\_n = \frac{1}{2\pi} \int^{\pi}\_{-\pi} f(t)e^{-int} dt $$. Combining this definition with the N-th partial sum, we get $$ S\_Nf= \sum^{N}\_{n = -N} \biggr(\frac{1}{2\pi} \int^{\pi}\_{-\pi} f(t)e^{-int} dt \biggr) e^{inx} = \frac{1}{2\pi} \int^{\pi}\_{-\pi} f(t) \biggr( \sum^{N}\_{n = -N} e^{in(x-t)} \biggr) dt $$.

From the above relation, we call the term in the final parentheses the Dirichlet kernel, written more generally as $D\_N(x) = \sum^{N}\_{n = -N} e^{inx} $.

$$ \frac{1}{2\pi}D\_N \* f = \frac{1}{2\pi}\int^{\pi}\_{-\pi} D\_N(x-t) f(t) dt = \frac{1}{2\pi}\int^{\pi}\_{-\pi} \biggr( \sum^{N}\_{n = -N} e^{in(x-t)} \biggr) \biggr(\sum^{N}\_{n = -N} c\_n e^{int} \biggr) dt = \frac{1}{2\pi}\int^{\pi}\_{-\pi}\sum^{N}\_{n = -N} c\_n e^{inx} dt = \sum^{N}\_{n = -N} c\_n e^{inx} = S\_Nf$$

Note how we obtain the definition of convolution when we use this identity. This shows us that the convolution of the Dirichlet kernel with any function $f$ of period $2 \pi$ is the N-th degree Fourier series approximation of $f$.

It’s easy to see that the definition of the Dirichlet kernel is a geometric series. Meaning that we can use the formula for a geometric series $$ \sum^{n}\_{k=0} ar^k = a\frac{1-r^{n+1}}{1-r}$$ to solve for it.

$$ \sum^{n}\_{k=-n} r^k = r^{-n}\frac{1-r^{2n+1}}{1-r} = \frac{r^{-n-\frac{1}{2}}}{r^{-\frac{1}{2}}}\frac{1-r^{2n+1}}{1-r} = \frac{r^{-n-\frac{1}{2}}-r^{-n-\frac{1}{2}}r^{2n+1}}{r^{-\frac{1}{2}}-r^{-\frac{1}{2}}r} = \frac{r^{-n-\frac{1}{2}}-r^{n+\frac{1}{2}}}{r^{-\frac{1}{2}}-r^{\frac{1}{2}}} $$

In this case, $r = e^{ix}$. Plugging that value in gives us

$$\sum^{N}\_{n = -N} e^{inx} = \frac{e^{-(N+\frac{1}{2})ix}-e^{(N+\frac{1}{2})ix}}{e^{-\frac{ix}{2}}-e^{\frac{ix}{2}}} = \frac{cos((N+\frac{1}{2})x)-isin((N+\frac{1}{2})x)-cos((N+\frac{1}{2})x)-isin((N+\frac{1}{2})x)}{cos(\frac{x}{2})-isin(\frac{x}{2})-cos(\frac{x}{2})-isin(\frac{x}{2})} = \frac{-2isin((N+\frac{1}{2})x)}{-2isin(\frac{x}{2})} = \frac{sin((N+\frac{1}{2})x)}{sin(\frac{x}{2})}$$

Which gives us the equation:

$$D\_N(x) = \frac{sin((N+\frac{1}{2})x)}{sin(\frac{x}{2})}$$

**Cesaro summation and the Fejer kernel**